

# QUARTET FIXED POINT THEOREMS FOR NONLINEAR CONTRACTIONS IN PARTIALLY ORDERED METRIC SPACES

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**ABSTRACT.** The notion of coupled fixed point is introduced in by Bhaskar and Lakshmikantham in [2]. Very recently, the concept of tripled fixed point is introduced by Berinde and Borcut [1]. In this manuscript, by using the mixed  $g$  monotone mapping, some new quartet fixed point theorems are obtained. We also give some examples to support our results.

## 1. INTRODUCTION AND PRELIMINARIES

In 2006, Bhaskar and Lakshmikantham [2] introduced the notion of coupled fixed point and proved some fixed point theorem under certain condition. Later, Lakshmikantham and Ćirić in [8] extended these results by defining of  $g$ -monotone property. After that many results appeared on coupled fixed point theory (see e.g. [3, 4, 6, 5, 11, 10]).

Very recently, Berinde and Borcut [1] introduced the concept of tripled fixed point and proved some related theorems. In this manuscript, the quartet fixed point is considered and by using the mixed  $g$ -monotone mapping, existence and uniqueness of quartet fixed point are obtained.

First we recall the basic definitions and results from which quartet fixed point is inspired. Let  $(X, d)$  be a metric space and  $X^2 := X \times X$ . Then the mapping  $\rho : X^2 \times X^2 \rightarrow [0, \infty)$  such that  $\rho((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + d(y_1, y_2)$  forms a metric on  $X^2$ . A sequence  $(\{x_n\}, \{y_n\}) \in X^2$  is said to be a double sequence of  $X$ .

**Definition 1.** (See [2]) Let  $(X, \leq)$  be partially ordered set and  $F : X \times X \rightarrow X$ .  $F$  is said to have mixed monotone property if  $F(x, y)$  is monotone nondecreasing in  $x$  and is monotone non-increasing in  $y$ , that is, for any  $x, y \in X$ ,

$$x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \quad \text{and}$$

$$y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1), \quad \text{for } y_1, y_2 \in X.$$

**Definition 2.** (see [2]) An element  $(x, y) \in X \times X$  is said to be a coupled fixed point of the mapping  $F : X \times X \rightarrow X$  if

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

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Throughout this paper, let  $(X, \leq)$  be partially ordered set and  $d$  be a metric on  $X$  such that  $(X, d)$  is a complete metric space. Further, the product spaces  $X \times X$  satisfy the following:

$$(u, v) \leq (x, y) \Leftrightarrow u \leq x, v \leq y; \text{ for all } (x, y), (u, v) \in X \times X. \quad (1.1)$$

The following two results of Bhaskar and Lakshmikantham in [2] were extended to class of cone metric spaces in [5]:

**Theorem 3.** *Let  $F : X \times X \rightarrow X$  be a continuous mapping having the mixed monotone property on  $X$ . Assume that there exists a  $k \in [0, 1)$  with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \text{ for all } u \leq x, y \leq v. \quad (1.2)$$

*If there exist  $x_0, y_0 \in X$  such that  $x_0 \leq F(x_0, y_0)$  and  $F(y_0, x_0) \leq y_0$ , then, there exist  $x, y \in X$  such that  $x = F(x, y)$  and  $y = F(y, x)$ .*

**Theorem 4.** *Let  $F : X \times X \rightarrow X$  be a mapping having the mixed monotone property on  $X$ . Suppose that  $X$  has the following properties:*

- (i) *if a non-decreasing sequence  $\{x_n\} \rightarrow x$ , then  $x_n \leq x, \forall n$ ;*
- (i) *if a non-increasing sequence  $\{y_n\} \rightarrow y$ , then  $y \leq y_n, \forall n$ .*

*Assume that there exists a  $k \in [0, 1)$  with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \text{ for all } u \leq x, y \leq v. \quad (1.3)$$

*If there exist  $x_0, y_0 \in X$  such that  $x_0 \leq F(x_0, y_0)$  and  $F(y_0, x_0) \leq y_0$ , then, there exist  $x, y \in X$  such that  $x = F(x, y)$  and  $y = F(y, x)$ .*

Inspired by Definition 1, the following concept of a  $g$ -mixed monotone mapping introduced by V. Lakshmikantham and L.Ćirić [8].

**Definition 5.** *Let  $(X, \leq)$  be partially ordered set and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$ .  $F$  is said to have mixed  $g$ -monotone property if  $F(x, y)$  is monotone  $g$ -non-decreasing in  $x$  and is monotone  $g$ -non-increasing in  $y$ , that is, for any  $x, y \in X$ ,*

$$g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y), \text{ for } x_1, x_2 \in X, \text{ and} \quad (1.4)$$

$$g(y_1) \leq g(y_2) \Rightarrow F(x, y_2) \leq F(x, y_1), \text{ for } y_1, y_2 \in X. \quad (1.5)$$

It is clear that Definition 13 reduces to Definition 9 when  $g$  is the identity.

**Definition 6.** *An element  $(x, y) \in X \times X$  is called a couple point of a mapping  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  if*

$$F(x, y) = g(x), \quad F(y, x) = g(y).$$

**Definition 7.** *Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  where  $X \neq \emptyset$ . The mappings  $F$  and  $g$  are said to commute if*

$$g(F(x, y)) = F(g(x), g(y)), \text{ for all } x, y \in X.$$

**Theorem 8.** *Let  $(X, \leq)$  be partially ordered set and  $(X, d)$  be a complete metric space and also  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  where  $X \neq \emptyset$ . Suppose that  $F$  has the mixed  $g$ -monotone property and that there exists a  $k \in [0, 1)$  with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} \left[ \frac{d(g(x), g(u)) + d(g(y), g(v))}{2} \right] \quad (1.6)$$

*for all  $x, y, u, v \in X$  for which  $g(x) \leq g(u)$  and  $g(v) \leq g(y)$ . Suppose  $F(X \times X) \subset g(X)$ ,  $g$  is sequentially continuous and commutes with  $F$  and also suppose either  $F$  is continuous or  $X$  has the following property:*

$$\text{if a non-decreasing sequence } \{x_n\} \rightarrow x, \text{ then } x_n \leq x, \text{ for all } n, \quad (1.7)$$

$$\text{if a non-increasing sequence } \{y_n\} \rightarrow y, \text{ then } y \leq y_n, \text{ for all } n. \quad (1.8)$$

*If there exist  $x_0, y_0 \in X$  such that  $g(x_0) \leq F(x_0, y_0)$  and  $g(y_0) \leq F(y_0, x_0)$ , then there exist  $x, y \in X$  such that  $g(x) = F(x, y)$  and  $g(y) = F(y, x)$ , that is,  $F$  and  $g$  have a couple coincidence.*

Berinde and Borcut [1] introduced the following partial order on the product space  $X^3 = X \times X \times X$ :

$$(u, v, w) \leq (x, y, z) \text{ if and only if } x \geq u, y \leq v, z \geq w, \quad (1.9)$$

where  $(u, v, w), (x, y, z) \in X^3$ . Regarding this partial order, we state the definition of the following mapping.

**Definition 9.** (See [1]) *Let  $(X, \leq)$  be partially ordered set and  $F : X^3 \rightarrow X$ . We say that  $F$  has the mixed monotone property if  $F(x, y, z)$  is monotone non-decreasing in  $x$  and  $z$ , and it is monotone non-increasing in  $y$ , that is, for any  $x, y, z \in X$*

$$\begin{aligned} x_1, x_2 \in X, x_1 \leq x_2 &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, y_1 \leq y_2 &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X, z_1 \leq z_2 &\Rightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned} \quad (1.10)$$

**Theorem 10.** (See [1]) *Let  $(X, \leq)$  be partially ordered set and  $(X, d)$  be a complete metric space. Let  $F : X \times X \times X \rightarrow X$  be a mapping having the mixed monotone property on  $X$ . Assume that there exist constants  $a, b, c \in [0, 1)$  such that  $a + b + c < 1$  for which*

$$d(F(x, y, z), F(u, v, w)) \leq ad(x, u) + bd(y, v) + cd(z, w) \quad (1.11)$$

*for all  $x \geq u, y \leq v, z \geq w$ . Assume that  $X$  has the following properties:*

- (i) *if non-decreasing sequence  $x_n \rightarrow x$ , then  $x_n \leq x$  for all  $n$ ,*
- (ii) *if non-increasing sequence  $y_n \rightarrow y$ , then  $y_n \geq y$  for all  $n$ ,*

*If there exist  $x_0, y_0, z_0 \in X$  such that*

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(x_0, y_0, z_0)$$

*then there exist  $x, y, z \in X$  such that*

$$F(x, y, z) = x \text{ and } F(y, x, y) = y \text{ and } F(z, y, x) = z$$

The aim of this paper is introduce the concept of quartet fixed point and prove the related fixed point theorems.

## 2. QUARTET FIXED POINT THEOREMS

Let  $(X, \leq)$  be partially ordered set and  $(X, d)$  be a complete metric space. We state the definition of the following mapping. Throughout the manuscript we denote  $X \times X \times X \times X$  by  $X^4$ .

**Definition 11.** (See [7]) Let  $(X, \leq)$  be partially ordered set and  $F : X^4 \rightarrow X$ . We say that  $F$  has the mixed monotone property if  $F(x, y, z, w)$  is monotone non-decreasing in  $x$  and  $z$ , and it is monotone non-increasing in  $y$  and  $w$ , that is, for any  $x, y, z, w \in X$

$$\begin{aligned} x_1, x_2 \in X, x_1 \leq x_2 &\Rightarrow F(x_1, y, z, w) \leq F(x_2, y, z, w), \\ y_1, y_2 \in X, y_1 \leq y_2 &\Rightarrow F(x, y_1, z, w) \geq F(x, y_2, z, w), \\ z_1, z_2 \in X, z_1 \leq z_2 &\Rightarrow F(x, y, z_1, w) \leq F(x, y, z_2, w), \\ w_1, w_2 \in X, w_1 \leq w_2 &\Rightarrow F(x, y, z, w_1) \geq F(x, y, z, w_2). \end{aligned} \quad (2.1)$$

**Definition 12.** (See [7]) An element  $(x, y, z, w) \in X^4$  is called a quartet fixed point of  $F : X \times X \times X \times X \rightarrow X$  if

$$\begin{aligned} F(x, y, z, w) &= x, & F(x, w, z, y) &= y, \\ F(z, y, x, w) &= z, & F(z, w, x, y) &= w. \end{aligned} \quad (2.2)$$

**Definition 13.** Let  $(X, \leq)$  be partially ordered set and  $F : X^4 \rightarrow X$ . We say that  $F$  has the mixed  $g$ -monotone property if  $F(x, y, z, w)$  is monotone  $g$ -non-decreasing in  $x$  and  $z$ , and it is monotone  $g$ -non-increasing in  $y$  and  $w$ , that is, for any  $x, y, z, w \in X$

$$\begin{aligned} x_1, x_2 \in X, g(x_1) \leq g(x_2) &\Rightarrow F(x_1, y, z, w) \leq F(x_2, y, z, w), \\ y_1, y_2 \in X, g(y_1) \leq g(y_2) &\Rightarrow F(x, y_1, z, w) \geq F(x, y_2, z, w), \\ z_1, z_2 \in X, g(z_1) \leq g(z_2) &\Rightarrow F(x, y, z_1, w) \leq F(x, y, z_2, w), \\ w_1, w_2 \in X, g(w_1) \leq g(w_2) &\Rightarrow F(x, y, z, w_1) \geq F(x, y, z, w_2). \end{aligned} \quad (2.3)$$

**Definition 14.** An element  $(x, y, z, w) \in X^4$  is called a quartet coincidence point of  $F : X^4 \rightarrow X$  and  $g : X \rightarrow X$  if

$$\begin{aligned} F(x, y, z, w) &= g(x), & F(y, z, w, x) &= g(y), \\ F(z, w, x, y) &= g(z), & F(w, x, y, z) &= g(w). \end{aligned} \quad (2.4)$$

Notice that if  $g$  is identity mapping, then Definition 13 and Definition 14 reduce to Definition 11 and Definition 12, respectively.

**Definition 15.** Let  $F : X^4 \rightarrow X$  and  $g : X \rightarrow X$ .  $F$  and  $g$  are called commutative if

$$g(F(x, y, z, w)) = F(g(x), g(y), g(z), g(w)), \quad \text{for all } x, y, z, w \in X. \quad (2.5)$$

For a metric space  $(X, d)$ , the function  $\rho : X^4 \times X^4 \rightarrow [0, \infty)$ , given by,

$$\rho((x, y, z, w), (u, v, r, t)) := d(x, u) + d(y, v) + d(z, r) + d(w, t)$$

forms a metric space on  $X^4$ , that is,  $(X^4, \rho)$  is a metric induced by  $(X, d)$ .

Let  $\Phi$  denote the all functions  $\phi : [0, \infty) \rightarrow [0, \infty)$  which is continuous and satisfy that

$$(i) \quad \phi(t) < t$$

(i)  $\lim_{r \rightarrow t+} \phi(r) < t$  for each  $r > 0$ .

The aim of this paper is to prove the following theorem.

**Theorem 16.** *Let  $(X, \leq)$  be partially ordered set and  $(X, d)$  be a complete metric space. Suppose  $F : X^4 \rightarrow X$  and there exists  $\phi \in \Phi$  such that  $F$  has the mixed  $g$ -monotone property and*

$$d(F(x, y, z, w), F(u, v, r, t)) \leq \phi \left( \frac{d(x, u) + d(y, v) + d(z, r) + d(w, t)}{4} \right) \quad (2.6)$$

for all  $x, u, y, v, z, r, w, t$  for which  $g(x) \leq g(u)$ ,  $g(y) \geq g(v)$ ,  $g(z) \leq g(r)$  and  $g(w) \geq g(t)$ . Suppose there exist  $x_0, y_0, z_0, w_0 \in X$  such that

$$\begin{aligned} g(x_0) &\leq F(x_0, y_0, z_0, w_0), & g(y_0) &\geq F(x_0, w_0, z_0, y_0), \\ g(z_0) &\leq F(z_0, y_0, x_0, w_0), & g(w_0) &\geq F(z_0, w_0, x_0, y_0). \end{aligned} \quad (2.7)$$

Assume also that  $F(X^4) \subset g(X)$  and  $g$  commutes with  $F$ . Suppose either

- (a)  $F$  is continuous, or
- (b)  $X$  has the following property:
  - (i) if non-decreasing sequence  $x_n \rightarrow x$ , then  $x_n \leq x$  for all  $n$ ,
  - (ii) if non-increasing sequence  $y_n \rightarrow y$ , then  $y_n \geq y$  for all  $n$ ,

then there exist  $x, y, z, w \in X$  such that

$$\begin{aligned} F(x, y, z, w) &= g(x), & F(x, w, z, y) &= g(y), \\ F(z, y, x, w) &= g(z), & F(z, w, x, y) &= g(w). \end{aligned}$$

that is,  $F$  and  $g$  have a common coincidence point.

*Proof.* Let  $x_0, y_0, z_0, w_0 \in X$  be such that (2.7). We construct the sequences  $\{x_n\}$ ,  $\{y_n\}$ ,  $\{z_n\}$  and  $\{w_n\}$  as follows

$$\begin{aligned} g(x_n) &= F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), \\ g(y_n) &= F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}), \\ g(z_n) &= F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}), \\ g(w_n) &= F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}). \end{aligned} \quad (2.8)$$

for  $n = 1, 2, 3, \dots$

We claim that

$$\begin{aligned} g(x_{n-1}) &\leq g(x_n), & g(y_{n-1}) &\geq g(y_n), \\ g(z_{n-1}) &\leq g(z_n), & g(w_{n-1}) &\geq g(w_n), \end{aligned} \quad \text{for all } n \geq 1. \quad (2.9)$$

Indeed, we shall use mathematical induction to prove (2.9). Due to (2.7), we have

$$\begin{aligned} g(x_0) &\leq F(x_0, y_0, z_0, w_0) = g(x_1), & g(y_0) &\geq F(x_0, w_0, z_0, y_0) = g(y_1), \\ g(z_0) &\leq F(z_0, y_0, x_0, w_0) = g(z_1), & g(w_0) &\geq F(z_0, w_0, x_0, y_0) = g(w_1). \end{aligned}$$

Thus, the inequalities in (2.9) hold for  $n = 1$ . Suppose now that the inequalities in (2.9) hold for some  $n \geq 1$ . By mixed  $g$ -monotone property of  $F$ , together with (2.8) and (2.3) we have

$$\begin{aligned} g(x_n) &= F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \leq F(x_n, y_n, z_n, w_n) = g(x_{n+1}), \\ g(y_n) &= F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}) \geq F(x_n, w_n, z_n, y_n) = g(y_{n+1}), \\ g(z_n) &= F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) \leq F(z_n, y_n, x_n, w_n) = g(z_{n+1}), \\ g(w_n) &= F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \geq F(z_n, w_n, x_n, y_n) = g(w_{n+1}), \end{aligned} \quad (2.10)$$

Thus, (2.9) holds for all  $n \geq 1$ . Hence, we have

$$\begin{aligned} \cdots g(x_n) &\geq g(x_{n-1}) \geq \cdots \geq g(x_1) \geq g(x_0), \\ \cdots g(y_n) &\leq g(y_{n-1}) \leq \cdots \leq g(y_1) \leq g(y_0), \\ \cdots g(z_n) &\geq g(z_{n-1}) \geq \cdots \geq g(z_1) \geq g(z_0), \\ \cdots g(w_n) &\leq g(w_{n-1}) \leq \cdots \leq g(w_1) \leq g(w_0), \end{aligned} \quad (2.11)$$

$$\text{Set } \delta_n = d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1})) + d(g(w_n), g(w_{n+1}))$$

We shall show that

$$\delta_{n+1} \leq 4\phi\left(\frac{\delta_n}{4}\right). \quad (2.12)$$

Due to (2.6), (2.8) and (2.11), we have

$$\begin{aligned} d(g(x_{n+1}), g(x_{n+2})) &= d(F(x_n, y_n, z_n, w_n), F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1})) \\ &\leq \phi\left(\frac{d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1})) + d(g(w_n), g(w_{n+1}))}{4}\right) \\ &\leq \phi\left(\frac{\delta_n}{4}\right) \end{aligned} \quad (2.13)$$

$$\begin{aligned} d(g(y_{n+1}), g(y_{n+2})) &= d(F(y_n, z_n, w_n, x_n), F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1})) \\ &\leq \phi\left(\frac{d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1})) + d(g(w_n), g(w_{n+1})) + d(g(x_n), g(x_{n+1}))}{4}\right) \\ &\leq \phi\left(\frac{\delta_n}{4}\right) \end{aligned} \quad (2.14)$$

$$\begin{aligned} d(g(z_{n+1}), g(z_{n+2})) &= d(F(z_n, w_n, x_n, y_n), F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1})) \\ &\leq \phi\left(\frac{d(g(z_n), g(z_{n+1})) + d(g(w_n), g(w_{n+1})) + d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))}{4}\right) \\ &\leq \phi\left(\frac{\delta_n}{4}\right) \end{aligned} \quad (2.15)$$

$$\begin{aligned} d(g(w_{n+1}), g(w_{n+2})) &= d(F(w_n, x_n, y_n, z_n), F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1})) \\ &\leq \phi\left(\frac{d(g(w_n), g(w_{n+1})) + d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1}))}{4}\right) \\ &\leq \phi\left(\frac{\delta_n}{4}\right) \end{aligned} \quad (2.16)$$

Due to (2.13)-(2.16), we conclude that

$$d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) + d(z_{n+1}, z_{n+2}) + d(w_{n+1}, w_{n+2}) \leq 4\phi\left(\frac{\delta_n}{4}\right) \quad (2.17)$$

Hence we have (2.12).

Since  $\phi(t) < t$  for all  $t > 0$ , then  $\delta_{n+1} \leq \delta_n$  for all  $n$ . Hence  $\{\delta_n\}$  is a non-increasing sequence. Since it is bounded below, there is some  $\delta \geq 0$  such that

$$\lim_{n \rightarrow \infty} \delta_n = \delta. \quad (2.18)$$

We shall show that  $\delta = 0$ . Suppose, to the contrary, that  $\delta > 0$ . Taking the limit as  $\delta_n \rightarrow \delta+$  of both sides of (2.12) and having in mind that we suppose  $\lim_{t \rightarrow r} \phi(r) < t$  for all  $t > 0$ , we have

$$\delta = \lim_{n \rightarrow \infty} \delta_{n+1} \leq \lim_{n \rightarrow \infty} 4\phi\left(\frac{\delta_n}{4}\right) = \lim_{\delta_n \rightarrow \delta+} 4\phi\left(\frac{\delta_n}{4}\right) < 4\frac{\delta}{4} < \delta \quad (2.19)$$

which is a contradiction. Thus,  $\delta = 0$ , that is,

$$\lim_{n \rightarrow \infty} [d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1}) + d(w_n, w_{n-1})] = 0. \quad (2.20)$$

Now, we shall prove that  $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}$  and  $\{g(w_n)\}$  are Cauchy sequences. Suppose, to the contrary, that at least one of  $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}$  and  $\{g(w_n)\}$  is not Cauchy. So, there exists an  $\varepsilon > 0$  for which we can find subsequences  $\{g(x_{n(k)})\}$ ,  $\{g(y_{n(k)})\}$  of  $\{g(x_n)\}$  and  $\{g(y_n)\}$ ,  $\{g(z_{n(k)})\}$  of  $\{g(z_n)\}$  and  $\{g(w_{n(k)})\}$  of  $\{g(w_n)\}$  with  $n(k) > m(k) \geq k$  such that

$$\begin{aligned} & d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) \\ & + d(g(z_{n(k)}), g(z_{m(k)})) + d(g(w_{n(k)}), g(w_{m(k)})) \geq \varepsilon. \end{aligned} \quad (2.21)$$

Additionally, corresponding to  $m(k)$ , we may choose  $n(k)$  such that it is the smallest integer satisfying (2.21) and  $n(k) > m(k) \geq k$ . Thus,

$$\begin{aligned} & d(g(x_{n(k)-1}), g(x_{m(k)})) + d(g(y_{n(k)-1}), g(y_{m(k)})) \\ & + d(g(z_{n(k)-1}), g(z_{m(k)})) + d(g(w_{n(k)-1}), g(w_{m(k)})) < \varepsilon. \end{aligned} \quad (2.22)$$

By using triangle inequality and having (2.21), (2.22) in mind

$$\begin{aligned} \varepsilon & \leq t_k = d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) \\ & + d(g(z_{n(k)}), g(z_{m(k)})) + d(g(w_{n(k)}), g(w_{m(k)})) \\ & \leq d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(x_{n(k)-1}), g(x_{m(k)})) \\ & + d(g(y_{n(k)}), g(y_{n(k)-1})) + d(g(y_{n(k)-1}), g(y_{m(k)})) \\ & + d(g(z_{n(k)}), g(z_{n(k)-1})) + d(g(z_{n(k)-1}), g(z_{m(k)})) \\ & + d(g(w_{n(k)}), g(w_{n(k)-1})) + d(g(w_{n(k)-1}), g(w_{m(k)})) \\ & < d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(y_{n(k)}), g(y_{n(k)-1})) + \\ & + d(g(z_{n(k)}), g(z_{n(k)-1})) + d(g(w_{n(k)}), g(w_{n(k)-1})) + \varepsilon. \end{aligned} \quad (2.23)$$

Letting  $k \rightarrow \infty$  in (2.23) and using (2.20)

$$\lim_{k \rightarrow \infty} t_k = \lim_{k \rightarrow \infty} \left[ \begin{aligned} & d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) \\ & + d(g(z_{n(k)}), g(z_{m(k)})) + d(g(w_{n(k)}), g(w_{m(k)})) \end{aligned} \right] = \varepsilon + \quad (2.24)$$

Again by triangle inequality,

$$\begin{aligned} t_k & = d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) \\ & + d(g(z_{n(k)}), g(z_{m(k)})) + d(g(w_{n(k)}), g(w_{m(k)})) \\ & \leq d(g(x_{n(k)}), g(x_{n(k)+1})) + d(g(x_{n(k)+1}), g(x_{m(k)+1})) + d(g(x_{m(k)+1}), g(x_{m(k)})) \\ & + d(g(y_{n(k)}), g(y_{n(k)+1})) + d(g(y_{n(k)+1}), g(y_{m(k)+1})) + d(g(y_{m(k)+1}), g(y_{m(k)})) \\ & + d(g(z_{n(k)}), g(z_{n(k)+1})) + d(g(z_{n(k)+1}), g(z_{m(k)+1})) + d(g(z_{m(k)+1}), g(z_{m(k)})) \\ & + d(g(w_{n(k)}), g(w_{n(k)+1})) + d(g(w_{n(k)+1}), g(w_{m(k)+1})) + d(g(w_{m(k)+1}), g(w_{m(k)})) \\ & \leq \delta_{m(k)+1} + \delta_{m(k)+1} + d(g(x_{n(k)+1}), g(x_{m(k)+1})) + d(g(y_{n(k)+1}), g(y_{m(k)+1})) \\ & + d(g(z_{n(k)+1}), g(z_{m(k)+1})) + d(g(w_{n(k)+1}), g(w_{m(k)+1})) \end{aligned} \quad (2.25)$$

Since  $n(k) > m(k)$ , then

$$\begin{aligned} & g(x_{n(k)}) \geq g(x_{m(k)}) \text{ and } g(y_{n(k)}) \leq g(y_{m(k)}), \\ & g(z_{n(k)}) \geq g(z_{m(k)}) \text{ and } g(w_{n(k)}) \leq g(w_{m(k)}). \end{aligned} \quad (2.26)$$

Hence from (2.26), (2.8) and (2.6), we have,

$$\begin{aligned}
d(g(x_{n(k)+1}), g(x_{m(k)+1})) &= d(F(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)})) \\
&\leq \phi \left( \begin{array}{l} \frac{1}{4}[d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))] \\ + d(g(z_{n(k)}), g(z_{m(k)})) + d(g(w_{n(k)}), g(w_{m(k)}))] \end{array} \right)
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
d(g(y_{n(k)+1}), g(y_{m(k)+1})) &= d(F(y_{n(k)}, z_{n(k)}, w_{n(k)}, x_{n(k)}), F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)})) \\
&\leq \phi \left( \begin{array}{l} \frac{1}{4}[d(g(y_{n(k)}), g(y_{m(k)})) + d(g(z_{n(k)}), g(z_{m(k)}))] \\ + d(g(w_{n(k)}), g(w_{m(k)})) + d(g(x_{n(k)}), g(x_{m(k)}))] \end{array} \right)
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
d(g(z_{n(k)+1}), g(z_{m(k)+1})) &= d(F(z_{n(k)}, w_{n(k)}, x_{n(k)}, y_{n(k)}), F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)})) \\
&\leq \phi \left( \begin{array}{l} \frac{1}{4}[d(g(z_{n(k)}), g(z_{m(k)})) + d(g(w_{n(k)}), g(w_{m(k)}))] \\ + d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))] \end{array} \right)
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
d(g(w_{n(k)+1}), g(w_{m(k)+1})) &= d(F(w_{n(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)}), F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})) \\
&\leq \phi \left( \begin{array}{l} \frac{1}{4}[d(g(w_{n(k)}), g(w_{m(k)})) + d(g(x_{n(k)}), g(x_{m(k)}))] \\ + d(g(y_{n(k)}), g(y_{m(k)})) + d(g(z_{n(k)}), g(z_{m(k)}))] \end{array} \right)
\end{aligned} \tag{2.30}$$

Combining (2.25) with (2.27)-(2.30), we obtain that

$$\begin{aligned}
t_k &\leq \delta_{n(k)+1} + \delta_{m(k)+1} + d(g(x_{n(k)+1}), g(x_{m(k)+1})) + d(g(y_{n(k)+1}), g(y_{m(k)+1})) \\
&\quad + d(g(z_{n(k)+1}), g(z_{m(k)+1})) + d(g(w_{n(k)+1}), g(w_{m(k)+1})) \\
&\leq \delta_{n(k)+1} + \delta_{m(k)+1} + t_k + 4\phi\left(\frac{t_k}{4}\right) \\
&< \delta_{n(k)+1} + \delta_{m(k)+1} + t_k + 4\frac{t_k}{4}
\end{aligned} \tag{2.31}$$

Letting  $k \rightarrow \infty$ , we get a contradiction. This shows that  $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}$  and  $\{g(w_n)\}$  are Cauchy sequences. Since  $X$  is complete metric space, there exists  $x, y, z, w \in X$  such that

$$\begin{aligned}
\lim_{n \rightarrow \infty} g(x_n) &= x \text{ and } \lim_{n \rightarrow \infty} g(y_n) = y, \\
\lim_{n \rightarrow \infty} g(z_n) &= z \text{ and } \lim_{n \rightarrow \infty} g(w_n) = w.
\end{aligned} \tag{2.32}$$

Since  $g$  is continuous, (2.32) implies that

$$\begin{aligned}
\lim_{n \rightarrow \infty} g(g(x_n)) &= g(x) \text{ and } \lim_{n \rightarrow \infty} g(g(y_n)) = g(y), \\
\lim_{n \rightarrow \infty} g(g(z_n)) &= g(z) \text{ and } \lim_{n \rightarrow \infty} g(g(w_n)) = g(w).
\end{aligned} \tag{2.33}$$

From (2.10) and by regarding commutativity of  $F$  and  $g$ ,

$$\begin{aligned}
g(g(x_{n+1})) &= g(F(x_n, y_n, z_n, w_n)) = F(g(x_n), g(y_n), g(z_n), g(w_n)), \\
g(g(y_{n+1})) &= g(F(x_n, w_n, z_n, y_n)) = F(g(x_n), g(w_n), g(z_n), g(y_n)), \\
g(g(z_{n+1})) &= g(F(z_n, y_n, x_n, w_n)) = F(g(z_n), g(y_n), g(x_n), g(w_n)), \\
g(g(w_{n+1})) &= g(F(z_n, w_n, x_n, y_n)) = F(g(z_n), g(w_n), g(x_n), g(y_n)),
\end{aligned} \tag{2.34}$$

We shall show that

$$\begin{aligned}
F(x, y, z, w) &= g(x), & F(x, w, z, y) &= g(y), \\
F(z, y, x, w) &= g(z), & F(z, w, x, y) &= g(w).
\end{aligned}$$

Suppose now (a) holds. Then by (2.8), (2.34) and (2.32), we have

$$\begin{aligned}
 g(x) &= \lim_{n \rightarrow \infty} g(g(x_{n+1})) = \lim_{n \rightarrow \infty} g(F(x_n, y_n, z_n, w_n)) \\
 &= \lim_{n \rightarrow \infty} F(g(x_n), g(y_n), g(z_n), g(w_n)) \\
 &= F(\lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(y_n), \lim_{n \rightarrow \infty} g(z_n), \lim_{n \rightarrow \infty} g(w_n)) \\
 &= F(x, y, z, w)
 \end{aligned} \tag{2.35}$$

Analogously, we also observe that

$$\begin{aligned}
 g(y) &= \lim_{n \rightarrow \infty} g(g(y_{n+1})) = \lim_{n \rightarrow \infty} g(F(x_n, w_n, z_n, y_n)) \\
 &= \lim_{n \rightarrow \infty} F(g(x_n), g(w_n), g(z_n), g(y_n)) \\
 &= F(\lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(w_n), \lim_{n \rightarrow \infty} g(z_n), \lim_{n \rightarrow \infty} g(y_n)) \\
 &= F(x, w, z, y)
 \end{aligned} \tag{2.36}$$

$$\begin{aligned}
 g(z) &= \lim_{n \rightarrow \infty} g(g(z_{n+1})) = \lim_{n \rightarrow \infty} g(F(z_n, y_n, x_n, w_n)) \\
 &= \lim_{n \rightarrow \infty} F(g(z_n), g(y_n), g(x_n), g(w_n)) \\
 &= F(\lim_{n \rightarrow \infty} g(z_n), \lim_{n \rightarrow \infty} g(y_n), \lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(w_n)) \\
 &= F(z, y, x, w)
 \end{aligned} \tag{2.37}$$

$$\begin{aligned}
 g(w) &= \lim_{n \rightarrow \infty} g(g(w_{n+1})) = \lim_{n \rightarrow \infty} g(F(z_n, w_n, x_n, y_n)) \\
 &= \lim_{n \rightarrow \infty} F(g(z_n), g(w_n), g(x_n), g(y_n)) \\
 &= F(\lim_{n \rightarrow \infty} g(z_n), \lim_{n \rightarrow \infty} g(w_n), \lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(y_n)) \\
 &= F(z, w, x, y)
 \end{aligned} \tag{2.38}$$

Thus, we have

$$\begin{aligned}
 F(x, y, z, w) &= g(x), & F(y, z, w, x) &= g(y), \\
 F(z, w, x, y) &= g(z), & F(w, x, y, z) &= g(w).
 \end{aligned}$$

Suppose now the assumption (b) holds. Since  $\{g(x_n)\}$ ,  $\{g(z_n)\}$  is non-decreasing and  $g(x_n) \rightarrow x$ ,  $g(z_n) \rightarrow z$  and also  $\{g(y_n)\}$ ,  $\{g(w_n)\}$  is non-increasing and  $g(y_n) \rightarrow y$ ,  $g(w_n) \rightarrow w$ , then by assumption (b) we have

$$g(x_n) \geq x, \quad g(y_n) \leq y, \quad g(z_n) \geq z, \quad g(w_n) \leq w \tag{2.39}$$

for all  $n$ . Thus, by triangle inequality and (2.34)

$$\begin{aligned}
 d(g(x), F(x, y, z, w)) &\leq d(g(x), g(g(x_{n+1}))) + d(g(g(x_{n+1})), F(x, y, z, w)) \\
 &\leq d(g(x), g(g(x_{n+1}))) + \phi \left( \frac{1}{4} \left[ \begin{aligned} &d(g(g(x_n), g(x))) + d(g(g(y_n), g(y))) \\ &+ d(g(g(z_n), g(z))) + d(g(g(w_n), g(w))) \end{aligned} \right] \right)
 \end{aligned} \tag{2.40}$$

Letting  $n \rightarrow \infty$  implies that  $d(g(x), F(x, y, z, w)) \leq 0$ . Hence,  $g(x) = F(x, y, z, w)$ . Analogously we can get that

$$F(y, z, w, x) = g(y), \quad F(z, w, x, y) = g(z) \quad \text{and} \quad F(w, x, y, z) = g(w).$$

Thus, we proved that  $F$  and  $g$  have a quartet coincidence point.  $\square$

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